Questions on special functions

I) Verify the following formulas

1) $$\int_0^\infty \frac{t^{c-1}}{(t+b)(a-t)}dt = \frac{\pi}{a+b}[b^{c-1}\csc c\pi + a^{c-1}\cot c\pi]$$

2) $$\int_0^\infty \frac{(t+b)t^{d-1}}{(t+a)(t+c)}dt = \frac{\pi}{\sin d\pi}[\frac{a-b}{a-c}a^{d-1} + \frac{c-b}{c-a}c^{d-1}]$$

3) $$\int_0^\infty \frac{t^{c+1}}{(1+t^2)^2}dt = \frac{c\pi}{4\sin(c\pi/2)}$$

4) $$\int_0^\infty \frac{t^{ac-1}}{(1+t^c)^{a+b}}dt = \frac{1}{c}\beta(a,b)$$

5) $$\int_1^\infty \frac{dt}{(a-bt)(t-1)^c} = -\frac{\pi}{b}[\frac{b}{b-a}]^c \csc c\pi$$

6) $$\int_{-\infty}^a \frac{(a-t)^{p-1}}{t-b}dt = -\frac{\pi}{\sin p\pi}[b-a]^{p-1}$$

7) $$\int_1^\infty \frac{(t-1)^a}{t^b}dt = \beta(a+1, b-a-1)$$

8) $$\int_0^\infty \frac{t^{a-1}}{(1+ut)^{p+1}}dt = \frac{1}{u^a}\beta(a, p+1-a)$$

9) $$\int_0^1 t^{aq-1}(1-t^q)^{b-1}dt = \frac{1}{q}\beta(a,b)$$

10) $\int_0^1 t^{p+q-1}(1-t^q)^{-p/q}dt = \frac{1}{q}\beta(1+p/q,1-p/q)$

11) $\int_0^1 t^{q/p-1}(1-t^q)^{-1/p}dt = \frac{1}{q}\beta(1/p,1-1/p)$

12) $\int_0^1 \frac{t^{aq-1}}{\sqrt[q]{(1-t^q)}}dt = \frac{1}{q}\beta(a,1-1/q)$

13) $\int_0^\infty \frac{t^{a-1}}{t+c}dt = \frac{\pi}{\tan(a\pi)}(-c)^{a-1}$

14) $\int_0^1 \frac{t^{3c-m}}{\sqrt[3]{(1-t^3)}}dt = \frac{1}{3}\beta(c+\frac{1-m}{3},\frac{2}{3})$

15) $\int_0^\infty \frac{a\,dt}{\sqrt{t}(a^2+t^2)} = \frac{\pi}{\sqrt{2a}}$

16) $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}dx = \frac{1}{a^n b^m}\beta(m,n)$ and hence find the value

$\int_0^\infty \frac{x^5}{(2+3x)^{16}}dx$

17) $\int_0^\infty x\ e^{-ax}\ \sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$ [Hint: put sinbx = Im. (e^{ibx})]

18) $\int_0^\infty x^{m-1}\cos ax\ dx = \frac{\Gamma m}{a^m}(\cos\frac{m\pi}{2})$ [Hint: put cosax = Re. (e^{iax})]

19) $\int_0^\infty x^{n-1} e^{-ax} \sin bx \, dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}} \sin(n \tan^{-1}(\frac{b}{a}))$ [Hint: put sinbx = Im. (e^{ibx})]

20) $\int_a^b (x-a)^m (b-x)^n dx = (b-a)^{m+n+1} \beta(m+1, n+1)$, then deduce that:

$\int_5^9 \sqrt[4]{(x-5)(9-x)} dx = \frac{2(\Gamma 1/4)^2}{3\sqrt{\pi}}$ [Hint : put x-a = (b-a)t]

21) $\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{\pi}{n} \csc(\frac{\pi}{n})$

22) $\int_0^{\pi/2} (\tan x)^n dx = \frac{\pi}{2} \sec(\frac{n\pi}{2})$

23) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{(\Gamma 1/4)^2}{4\sqrt{\pi}}$ [Hint: put tan(x/2) = t]

24) $\int_0^{\pi/2} (\sin x)^{2n} dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma[(n+1)/2]}{n!}$

25) $\int_0^{\pi/2} (\sin\theta)^P d\theta = \int_0^{\pi/2} (\cos\theta)^P d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2}+1)}$

II) Evaluate the following integrals

24) $\int_0^2 \frac{t^2 \, dt}{\sqrt{2-t}}$

25) $\int_{-1}^1 (1-t^2)^n dt$,

26) $\int_0^{1/2} t^{m-1} \ln(1/2t)\, dt$,

27) $\int_0^{\infty} \frac{t^2\, dt}{1+t^4}$,

28) $\int_0^{\pi} \frac{dt}{\sqrt{3-\cos t}} = \frac{\Gamma(1/4)}{4\sqrt{\pi}}$ (Hint: put cos t = 1 - $2\sqrt{y}$)

29) $\int_0^{\pi/2} \cos^m t \sin^n t\, dt$

30) $\int_0^{3} \frac{dt}{\sqrt{3t-t^2}}$

31) $\int_0^{1} t^3 \sqrt[3]{8-t^3}\, dt$

32) $\int_0^{1} \frac{dt}{\sqrt{-\mathrm{Ln}\, t}}$

33) $\int_0^{\infty} t^n e^{-mt} dt$

34) $\int_0^{\infty} x^n e^{-\sqrt{ax}} dx$

35) $\int_0^{\infty} \frac{x^n\, dx}{a^x}$

36) $\int_0^{\infty} a^{-m x^n} dx$

37) $\int_0^{\infty} \sqrt{x}\, e^{-\sqrt[3]{x}} dx$

38) $\int_0^\infty \frac{e^{-\sqrt{x}}\,dx}{x^{7/4}}$

39) $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx$

40) $\int_0^{\pi/4} (\cos 2\theta)^3 (\sin 4\theta)^4 d\theta$ [Hint: put $2\theta = t$]

41) $\int_0^{2\pi} (\sin\theta)^2 (1+\cos\theta)^4 d\theta$ [Hint: put $\theta/2 = t$]

42) $\int_{-\pi/4}^{\pi/4} (\sin\theta + \cos\theta)^{1/3} d\theta$ [Hint:

$\int_{-\pi/4}^{\pi/4} [\sqrt{2}(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta)]^{1/3} d\theta = \int_{-\pi/4}^{\pi/4} [\sqrt{2}\cos(\theta + \pi/4)]^{1/3} d\theta$, put

$\theta + \pi/4 = t$]

43) $\int_{-\pi/6}^{\pi/3} (\sqrt{3}\sin\theta + \cos\theta)^{1/4} d\theta$

44) $\int_0^\infty \frac{x^8(1-x^6)\,dx}{(1+x)^{24}}$ [Hint: $\int_0^\infty \frac{x^8(1-x^6)\,dx}{(1+x)^{24}} =$

$\int_0^\infty \frac{(x^8 - x^{14})\,dx}{(1+x)^{24}}$]

45) $\int_0^\pi \frac{(\sin x)^{n-1}\,dx}{a + b\cos x}$ [Hint: put $\tan(x/2) = t$]

46) Given $\Gamma\left(\frac{8}{5}\right) = 0.8935$, find the value of $\Gamma\left(\frac{-12}{5}\right)$

47) If $\beta(n,2)=\frac{1}{42}$ and n is positive integer, find the value of n.

48) $\int_{-a}^{a}(a+x)^{m-1}(a-x)^{n-1}dx$

49) $\int_{0}^{1} x^m \, (\log_a x)^n \, dx$

50) $\int_{0}^{1} (-\log_a x)^{-1/2} \, dx$

51) $\int_{0}^{\infty} x^m \, e^{-ax^n} \, dx$

52) $\int_{0}^{2} x\sqrt{(8-x^3)}dx$

III) Choose the correct answer

48) $\int_{0}^{1} x^4 \, (\log \frac{1}{x})^3 \, dx$ = (3/325, 6/625, 3/625, 6/325)

49) $\int_{0}^{\pi/2} \sqrt{\cot x}dx = (\sqrt{2}\pi/2, \pi/2, \pi/4, \sqrt{2}\pi/4)$

50) $\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-x^2/8} dx = (1, 2, \pi, 2\pi)$

51) $\int_{0}^{\infty} \sqrt{y} \, e^{-y^3} dy = (\sqrt{\pi}/2, \sqrt{\pi}/3, \sqrt{\pi}, \sqrt{\pi}/6)$

52) If $\beta(n,2)=\frac{1}{6}$ and n is positive integer, then the value of n is (3, -2, 2, -3)

53) The value of $\beta(m+1,n)$ is

$$\left(\frac{n}{m+n}\beta(m,n),\ \frac{n}{m+1}\beta(m,n),\ \frac{m}{m+n}\beta(m,n),\ \frac{m}{m+1}\beta(m,n)\right)$$

54) $\int_0^\infty \frac{t^2}{1+t^4}dt = (\pi/\sqrt{2},\ \sqrt{\pi}/2,\ \pi/2,\ \pi/4)$

55) $\int_0^\infty e^{-y^3}dy = \left(\frac{1}{3}\left|\frac{1}{3}\right.,\ \frac{1}{3}\left|\frac{1}{2}\right.,\ \frac{1}{9}\left|\frac{1}{3}\right.,\ \frac{1}{3}\left|\frac{2}{3}\right.\right)$

56) $\int_0^1 x^2(1-x)^{5/2}dx = (\beta(3,7/2),\ \beta(1,5/2),\ \beta(2,5/2),\ \beta(1,3/2))$

57) $\int_0^\infty x^4 e^{-x^2}dx = (3\sqrt{\pi}/8,\ 3\sqrt{\pi}/4,\ \sqrt{\pi}/2,\ 15\sqrt{\pi}/8)$

58) $\int_0^1 \frac{dx}{\sqrt{1-x^4}} =$

$$\left(\Gamma(3/4)\Gamma(1/2),\ \Gamma(3/4)\Gamma(1/4),\ \frac{(1/4)\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)},\ \frac{(1/2)\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)}\right)$$

59) Prove that :

$$\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}$$

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$$

60) Match the items in columns I and II

a) $\beta(p,q)$	i) $\Gamma(1/2)$
b) $\dfrac{\Gamma p \Gamma q}{\Gamma(p+q)}$	ii) $\int_0^\infty \dfrac{x^{p-1}dx}{(1+x)^{p+q}}$
c) $\sqrt{\pi}$	iii) $\beta(p,q)$
d) $\dfrac{\pi}{\sin p\pi}$	iv) $\Gamma p \Gamma(1-p)$